On the edge universality of the local eigenvalue statistics of matrix models

L. Pastur[1,2], and M. Shcherbina[2]
[1] University Paris 7, Paris, France
[2] Institute for Low Temperature Physics, Kharkov, Ukraine

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Abstract

Basing on our recent results on the $1/n$-expansion in unitary invariant random matrix ensembles, known as matrix models, we prove that the local eigenvalue statistic, arising in a certain neighborhood of the edges of the support of the Density of States, is independent of the form of the potential, determining the matrix model. Our proof is applicable to the case of real analytic potentials and of supports, consisting of one or two disjoint intervals.

1 Introduction

Universality is an important concept of the random matrix theory and of its numerous applications (see e.g. reviews [14, 19] and references therein). In a more concrete context one refers to universality while dealing with local eigenvalue statistics of ensembles of $n \times n$ real symmetric, Hermitian or real quaternion matrices in the limit $n \to \infty$. One distinguishes the bulk case, arising in a $1/n$-neighborhood of a point λ_0 of the support of the Density of States ρ of an ensemble, such that $0 < \rho(\lambda_0) < \infty$, and the edge case, arising in a certain $o(1)$-neighborhood of endpoints of the support, more generally, in a neighborhood of those points of the support, at which $\rho(\lambda_0) = 0, \infty$ (perhaps as an one-side limit, i.e., for $\lambda_0 + 0$ or $\lambda_0 - 0$).

In this paper we will study the edge universality of ensembles of Hermitian matrices $M = \{M_{jk}\}_{j,k=1}^n$, $\overline{M}_{jk} = M_{j,k}$, known as the matrix models and defined by the probability distribution

$$P_n(M)dM = Z_n^{-1} \exp\{-n\mathrm{Tr}V(M)\}dM, \tag{1.1}$$

where

$$dM = \prod_{j=1}^n dM_{jj} \prod_{1 \le j < k \le n} d\Im M_{jk} d\Re M_{jk}, \tag{1.2}$$

Z_n is the normalizing constant, and the function $V : \mathbf{R} \to \mathbf{R}_+$ is called the potential. We assume that V satisfies the conditions:

(i) there exist L_1 and $\epsilon > 0$, such that

$$|V(\lambda)| \ge (2 + \epsilon) \log |\lambda|, \ |\lambda| \ge L_1, \tag{1.3}$$

(ii) for any $0 < L_2 < \infty$ and some $\gamma > 0$

$$|V(\lambda_1) - V(\lambda_2)| \le C(L_2)|\lambda_1 - \lambda_2|^\gamma, \ \ |\lambda_{1,2}| \le L_2. \tag{1.4}$$

Denote by $\lambda_1^{(n)}, ..., \lambda_n^{(n)}$ the eigenvalues of a matrix M_n and define its Eigenvalue Counting Measure as

$$N_n(\Delta) = \sharp\{\lambda_l^{(n)} \in \Delta, \ l = 1, \ldots, n\} \cdot n^{-1}, \tag{1.5}$$

where Δ is an interval of the spectral axis. According to [7, 15] the N_n tends weakly in probability as $n \to \infty$ to the non-random measure N known as the Integrated Density of States (IDS) of the ensemble. The measure N is a unique minimizer of the functional

$$\mathcal{E}[m] = \int V(\lambda)m(d\lambda) - \int \int \log|\lambda - \mu|m(d\lambda)m(d\mu), \tag{1.6}$$

defined on non-negative unit measures on $\mathbf{R}$. Here and below integrals without limits denote the integration over the whole real axis.

The IDS N is normalized to unity and is absolutely continuous if V' satisfies the Lipshitz condition [22]:

$$N(\mathbf{R}) = 1, \ N(\Delta) = \int_\Delta \rho(\lambda)d\lambda. \tag{1.7}$$

The non-negative function ρ in (1.7) is called the Density of States (DOS) of the ensemble. The DOS of matrix models was studied in [7, 15, 9]. It follows from these papers that in the case of a real analytic potential the support of the DOS consists of a finite number of finite disjoint intervals and that if a_* is an endpoint of the support, then the DOS behaves asymptotically as $\rho(\lambda) = \mathrm{const} \cdot |\lambda - a_*|^{1/2}, \ \lambda \to a_*$ generically in V.

The most studied ensemble of the random matrix theory is the Gaussian Unitary Ensemble, determined by (1.1) - (1.2) with

$$V(\lambda) = 2\lambda^2/a^2. \tag{1.8}$$

Here the DOS is the semi-circle low of Wigner

$$\rho(\lambda) = \frac{2}{\pi a^2}(a^2 - \lambda^2)_+^{1/2}, \tag{1.9}$$

where $x_+ = \max(x, 0)$.

The most known quantity probing the universality is the large-n form of the hole probability

$$E_n(\Delta_n) = \mathbf{P}_n\{\lambda_l^{(n)} \notin \Delta_n, \ l = 1, ..., n\}, \tag{1.10}$$

where $\mathbf{P}_n\{...\}$ is the probability defined by the distribution (1.1) - (1.2), and Δ_n is an interval of the spectral axis, whose order of magnitude is fixed by the condition $nN(\Delta_n)|\Delta_n| \approx 1$.

In the bulk case we choose [17]

$$\Delta_n = (\lambda_0, \lambda_0 + s/n\rho(\lambda_0)), \ s > 0, \ 0 < \rho(\lambda_0) < \infty. \tag{1.11}$$

In this case the limiting hole probability is the Fredholm determinant of the integral operator, defined by the kernel $\sin \pi(t_1 - t_2)/\pi(t_1 - t_2)$ on the interval $(0, s)$. This fact for the Gaussian Unitary Ensemble (1.8) was established by M. Gaudin in the early 60s [17]. The same fact was proved recently in [20, 10] for certain classes of matrix models. This is an example of bulk universality, showing that the local (in the sense (1.10) - (1.11)) statistical properties of eigenvalues do not depend on the ensemble, i.e., on the function V in (1.1), modulo a proper rescaling of the spectral axis.

The edge case of local eigenvalue statistics was studied much later even for the GUE [12, 23]. It was found that if we choose

$$\Delta_n = \left(a, a(1 + s/2n^{2/3}) \right), \ s \in \mathbf{R} \tag{1.12}$$

for the right-hand edge of the support $[-a, a]$ of (1.9), then the limit as $n \to \infty$ of the hole probability (1.10) of the GUE is the Fredholm determinant of the integral operator, defined on the interval $(0, s)$ by the kernel

$$\mathcal{K}(t_1, t_2) = \frac{Ai(t_1)Ai'(t_2) - Ai'(t_1)Ai(t_2)}{t_1 - t_2}, \tag{1.13}$$

where Ai is the standard Airy function [1]. Similar result is valid for the left-hand edge of the support of (1.9). Hence, the edge universality means that if a_* is an endpoint of the DOS support, and ρ behaves asymptotically as $\rho(\lambda) = \mathrm{const} \cdot |\lambda - a_*|^{1/2}, \ \lambda \to a_*$, then the limiting hole probability should be the same Fredholm determinant.

This fact for real analytic potentials in (1.1) can be deduced, under certain conditions, from the recent results [9] on the asymptotics of orthogonal polynomials on the whole line with the weight $e^{-nV(\lambda)}$. In this paper we give another proof of the edge universality of the eigenvalue statistics for the same class of potentials, assuming additionally that these potentials lead to the DOS, whose support

is either an interval $[a, b]$ or, in the case of even potentials, that the DOS support is $[-b, a] \cap [a, b]$, where $0 < a < b < \infty$. The proof is based on our recent results on the $1/n$-expansions for the matrix models [3], establishing, in particular, the "slow varying" character of the coefficients of the three-term recurrent relation (the finite-difference equation) for respective orthogonal polynomials. As a result, this relation becomes the Airy differential equation, leading to the kernel (1.13) in the interval (1.12). We believe that our proof makes explicit an important mathematical mechanism of the edge universality and is related to simplest cases of the double scaling limit in the matrix models of the Quantum Field Theory (see e.g. [8], for the random matrix content of these results).

2 Main Result

We will assume that the potential V, determining the probability law (1.1), satisfies the following conditions, in addition to conditions (1.3) and (1.4) above.

Condition C1. *The support σ of the IDS of the ensemble consists of either*

(i) *a single interval:*
$$\sigma = [a, b], \quad -\infty < a < b < \infty,$$

 or

(ii) *two symmetric intervals:*
$$\sigma = [-b, -a] \cup [a, b], \quad 0 < a < b < \infty, \tag{2.1}$$

 and V is even: $V(\lambda) = V(-\lambda)$, $\lambda \in \mathbf{R}$.

Remark. It is easy to see that changing the variables according to $M' = M - \dfrac{a+b}{2} I$ in case (i) we can always take the support σ to be symmetric with respect to the origin. Therefore without loss of generality we can assume that in this case

$$\sigma = [-a, a]. \tag{2.2}$$

Condition C2. *The DOS $\rho(\lambda)$ is strictly positive in the interior of the support σ and behaves asymptotically as $\mathrm{const} \cdot |\lambda - a_*|^{1/2}$, $\lambda \to a_*$, in a neighborhood of each edge a_* of the support. Besides, the function*

$$u(\lambda) = 2 \int \log |\mu - \lambda| \rho(\mu) d\mu - V(\lambda) \tag{2.3}$$

attains its maximum if and only if λ belongs to the interior of the closed set σ. We will call this behavior generic (see e.g. [16] for results, justifying the term)

Condition C3. *$V(\lambda)$ is real analytic on σ, i.e., there exists an open domain $\mathbf{D} \subset \mathbf{C}$ and an analytic in $\mathbf{D}$ function $\mathcal{V}(z), z \in \mathbf{D}$ such that*

$$\sigma \subset \mathbf{D}, \ \mathcal{V}(\lambda + i0) = V(\lambda), \quad \lambda \in \sigma.$$

We mention that we always have the single interval case if V is convex [7, 15] or if it has a unique absolute minimum and sufficiently large amplitude [16], and we always have the two interval case if V has two equal absolute minima and sufficiently large amplitude. Conditions C2 and C3 were used in paper [10] to obtain asymptotic formulas for orthogonal polynomials with the weight e^{-nV}. The condition C3 is the case in many applications of the random matrix theory to the Quantum Field Theory [13] and to the condensed matter theory [14, 5], where V is often a polynomial.

 The following statement known, in fact, in several contexts, provides a sufficiently explicit form of the DOS in our case (see e.g [3] for a proof).

Proposition 2.1 *Consider an ensemble of form (1.1)–(1.2), satisfying conditions (1.3), and C1–C3 above. Then its density of states (DOS) ρ has the form*

$$\rho(\lambda) = \frac{1}{2\pi} \chi_\sigma(\lambda) P(\lambda) X_+(\lambda), \tag{2.4}$$

where $\chi_\sigma(\lambda)$ is the indicator of the support σ of the DOS,

$$X_+(\lambda) = \begin{cases} \sqrt{a^2 - \lambda^2}, & |\lambda| \le a, \text{ in case } (2.2), \\ \text{sign } \lambda \cdot \sqrt{(\lambda^2 - a^2)(b^2 - \lambda^2)}, & a \le |\lambda| \le b, \text{ in case } (2.1), \end{cases} \tag{2.5}$$

and

$$P(z) = \frac{1}{\pi} \int_\sigma \frac{\mathcal{V}'(z) - V'(\lambda)}{z - \lambda} \frac{d\lambda}{X_+(\lambda)}. \tag{2.6}$$

Besides, the Stieltjes transform

$$g(z) = \int \frac{\rho(\mu) d\mu}{z - \mu}, \quad \Im z \neq 0, \tag{2.7}$$

of the DOS for $z \in \mathbf{D}$ satisfies the quadratic equation

$$g^2(z) - \mathcal{V}'(z)g(z) + \mathcal{Q}(z) = 0, \quad z \in \mathbf{D}, \tag{2.8}$$

where

$$\mathcal{Q}(z) = \int \frac{\mathcal{V}'(z) - V'(\lambda)}{z - \lambda} \rho(\lambda) d\lambda. \tag{2.9}$$

Denote by $p_n(\lambda_1, ..., \lambda_n)$ the joint eigenvalue probability density which we assume to be symmetric without loss of generality. It is known that [17]

$$p_n(\lambda_1, ... \lambda_n) = Q_n^{-1} \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2 \exp\left\{-n \sum_{l=1}^n V(\lambda_l)\right\}, \tag{2.10}$$

where Q_n is the respective normalization factor. Let

$$p_{l,n}(\lambda_1, ..., \lambda_l) = \int p_n(\lambda_1, ..., \lambda_l, \lambda_{l+1}, ... \lambda_n) d\lambda_{l+1} ... d\lambda_n \tag{2.11}$$

be the lth marginal distribution density of (2.10). Define the correlation functions as

$$\mathcal{R}_{l,n}(\lambda_1, ..., \lambda_l) = \frac{n!}{(n-l)!} p_l^{(n)}(\lambda_1, ..., \lambda_l). \tag{2.12}$$

The link with orthogonal polynomials is provided by the formulas [17, 6]

$$\mathcal{R}_{l,n}(\lambda_1, ..., \lambda_l) = \det\{K_n(\lambda_j, \lambda_k)\}_{j,k=1}^l, \tag{2.13}$$

$$E_n(\Delta_n) = \sum_{l=0}^{n-1} \frac{(-1)^l}{l!} \int_{\Delta_n^l} \det\{K_n(\lambda_j, \lambda_k)\}_{j,k=1}^l d\lambda_1 \ldots d\lambda_l. \tag{2.14}$$

Here

$$K_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu) \tag{2.15}$$

is known as the reproducing kernel of the orthonormalized system,

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\}p_l^{(n)}(\lambda), \ l = 0, ..., \tag{2.16}$$

in which $p_l^{(n)}$, $l = 0, ...$ are orthogonal polynomials on $\mathbf{R}$ associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)}, \tag{2.17}$$

i.e.,

$$\int p_l^{(n)}(\lambda)p_m^{(n)}(\lambda)w_n(\lambda)d\lambda = \delta_{l,m}. \tag{2.18}$$

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The polynomial $p_l^{(n)}$ has the degree l and a positive coefficient $\gamma_l^{(n)}$ in front of λ^l. The orthonormalized functions $\{\psi_l^{(n)}\}_{l \geq 0}$ of (2.16) verify the recurrent relations

$$J_l^{(n)}\psi_{l+1}^{(n)}(\lambda) + q_l^{(n)}\psi_l^{(n)}(\lambda) + J_{l-1}^{(n)}\psi_{l-1}^{(n)}(\lambda) = \lambda\psi_l^{(n)}(\lambda), \quad J_{-1}(n) = 0, \quad l = 0, \dots. \tag{2.19}$$

According to condition (1.3) the polynomials $p_l^{(n)}$ and the coefficients $J_l^{(n)}$ are defined for all l such that

$$l \leq n_1, \quad n_1 = n(1 + \epsilon/4). \tag{2.20}$$

In other words, we have here the $n_1 \times n_1$ real symmetric Jacobi matrix

$$\begin{aligned} J^{(n)} &= \{J_{l,m}^{(n)}\}_{l,m=0}^{n_1}, \\ J_{l,m}^{(n)} &= q_l^{(n)}\delta_{l,m} + J_l^{(n)}\delta_{l+1,m} + J_{l-1}^{(n)}\delta_{l-1,m}. \end{aligned} \tag{2.21}$$

Note that if V is even, then $q_l^{(n)} = 0$, $l = 0, \dots$.

We will need an important particular case of the above formulas [17], corresponding to $l = 1$ in (2.13):

$$\mathbf{E}_n\{N_n(\Delta)\} = \int_\Delta \rho_n(\lambda)d\lambda, \quad \rho_n(\lambda) = n^{-1}K_n(\lambda, \lambda), \tag{2.22}$$

where the symbol $\mathbf{E}_n\{\dots\}$ denotes the expectation with respect to the measure, defined by (2.10), i.e., by (1.1) - (1.2).

Theorem 2.2 *Consider an ensemble of the form (1.1) - (1.2), satisfying conditions (1.4), (1.3), and C1 - C3 above. Then for any endpoint a_* of the support σ and for any positive integer l the rescaled correlation function*

$$(\gamma n^{2/3})^{-l}\mathcal{R}_{l,n}(a_* \pm t_1/\gamma n^{2/3}, \dots, a_* \pm t_l/\gamma n^{2/3}) \tag{2.23}$$

converges weakly as $n \to \infty$ to

$$\det\{\mathcal{K}(t_j, t_k)\}_{j,k=1}^l, \tag{2.24}$$

where the sign $\pm$ (2.23) corresponds to a right hand and left hand endpoint, $\mathcal{K}(t_j, t_k)$ is the Airy kernel (1.13),

$$\gamma = (2c^2\alpha)^{-1/3} \tag{2.25}$$

and

$$\alpha = a, \quad c = \frac{1}{2a}\left(\frac{1}{P(a)} + \frac{1}{P(-a)}\right) \tag{2.26}$$

in the case (2.2) and

$$\alpha = (b^2 - a^2)\begin{cases} a^{-1}, \\ b^{-1}, \end{cases} \quad c = \begin{cases} \dfrac{2}{(b^2 - a^2)P(a)}, \\ \dfrac{2}{(b^2 - a^2)P(b)}, \end{cases} \tag{2.27}$$

in the case (2.1) for the endpoint a and b respectively. The function $P(\lambda)$, entering (2.26) and (2.27) is defined in (2.6).

Besides, if $\Delta \subset \mathbf{R}$ is a finite union of disjoint intervals bounded from the left in the case of the right hand endpoint and from the right in the case of the left hand endpoint, and $E_n(\Delta)$ is the hole probability (1.10) for $\Delta_n = a_* \pm \Delta/\gamma n^{2/3}$, then

$$\lim_{n \to \infty} E_n(\Delta_n) = 1 + \sum_{l=1}^\infty \frac{(-1)^l}{l!} \int_\Delta dt_1 \dots dt_l \det\{\mathcal{K}(t_j, t_k)\}_{j,k=1}^l, \tag{2.28}$$

i.e., the limit is the Fredholm determinant of the integral operator $\mathcal{K}_\Delta$, defined by the kernel (1.13) on the set Δ and the sign $\pm$ in Δ_n corresponds to the right and left hand endpoints of the support.

We mention now two particular cases of the theorem. The first corresponds to the case $l = 1$.

Corollary 2.3 *Denote*

$$\nu_n(s) = \rho_n(a_* \pm s/\gamma n^{2/3}) n^{1/3}/\gamma. \tag{2.29}$$

Then we have weakly in $\mathbf{R}$ *:*

$$\lim_{n\to\infty} \nu_n = \nu,$$

where

$$\nu(s) = \int_s^\infty Ai^2(x) dx, \tag{2.30}$$

and $\pm$ *in (2.29) corresponds to the right and left hand endpoints respectively.*

Remarks. 1). Denote $\mathcal{N}_n(\Delta)$ the number of eigenvalues in the set $\Delta_n = a_* \pm \Delta/\gamma n^{2/3}$, $\Delta \subset \mathbf{R}$. According to (2.22) and (2.29)

$$\mathbf{E}_n\{\mathcal{N}_n(\Delta)\} = n \int_{a_*+2cn^{-2/3}\Delta} \rho_n(\lambda) d\lambda = \int_\Delta \nu_n(s) ds.$$

Hence, we can interpret the corollary as a statement, according to which the expectation of the rescaled counting measure $\mathcal{N}_n$ converges weakly to the absolute continuous measure $\mathcal{N}$ whose density is (2.30). The density can be viewed as an analogue of the density of states for $n^{-2/3}$- neighborhoods of the edge a_* of the support of the Density of States, given by Proposition 2.1.

2). By using the equation

$$Ai''(x) = xAi(x), \tag{2.31}$$

and the following from the equation identity [1]

$$\frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y} = \int_0^\infty Ai(x+u)Ai(y+u) du \tag{2.32}$$

for $x = y$, we can rewrite the r.h.s. of (2.30) in the form

$$\nu(s) = Ai'^2(s) - sAi^2(s). \tag{2.33}$$

The formula was obtained in [12] in the case (1.8) of the GUE, by using (2.22) and the Plancherel-Rotah asymptotic formula for the Hermite polynomials, that play the role of polynomials $p_l^{(n)}$ for the GUE [17].

The next corollary deals with the case of $\Delta = (s, \infty)$ of formula (2.28).

Corollary 2.4 *Under condition of Theorem 2.2 the* $n = \infty$ *limit of the probability distribution of the maximum eigenvalue of the random matrix (1.1) - (1.2) is*

$$\lim_{n\to\infty} \mathbf{P}_n\{\lambda_{\max}^{(n)} \le a_* + s/\gamma n^{2/3}\} = \det(1 - \mathcal{K}(s)),$$

where a_* *is the extreme right-hand endpoint of the support and* $\mathcal{K}(s)$ *is the integral operator, defined by the kernel (1.13) on the interval* (s, ∞).

The corollary asserts that the $n = \infty$ limit of the probability distribution of the maximum eigenvalue of the random matrix is independent of the ensemble (of function V in 1.1), i.e., the universality of this distribution for the class of ensembles, treated in the paper. Analogous statement is valid for the minimum eigenvalue.

3 Proof of the Main Result

We will prove Theorem 2.2 in details for the case (i), where the support of the Density of States is an interval of the spectral axis. At the end of the proof we shall explain the difference between this case and the two-interval case (ii). Besides, we can restrict ourselves to the right hand endpoint a of the interval $[-a, a]$ without loss of generality.

The proof is based on the following asymptotic formula for the coefficient $J_l^{(n)}$ of the Jacobi matrix (2.21) [3], Theorem 1:

$$J_{n+k}^{(n)} = \frac{a}{2} + \frac{k}{n}c + r_k^{(n)}, \quad q_{n+k} = \overline{r}_k^{(n)}, \tag{3.1}$$

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where c is given by (2.26), and the remainders $r_k^{(n)}$, $\overline{r}_k^{(n)}$ admit the estimate

$$|r_k^{(n)}|, |\overline{r}_k^{(n)}| \leq C \cdot \frac{k^2+1}{n^2}, \quad |k| \leq C \cdot n^{2/3}. \tag{3.2}$$

Here and below the symbol C denotes positive quantities that do not depend on n and k but can be different in different formulas.

It follows from (2.19) and the orthonormality of $\{p_l^{(n)}\}_{l \geq 0}$ that

$$J_l^{(n)} = \int \lambda p_{l+1}^{(n)}(\lambda) p_l^{(n)}(\lambda) d\lambda. \tag{3.3}$$

This and (3.1) imply that the order of the orthogonal polynomials $p_l^{(n)}$, entering formulas (2.13) - (2.15), and (3.1) does not exceed $n + Cn^{2/3}$, and makes possible to replace $p_l^{(n)}$ of (2.18), orthogonal on the whole axis, by the polynomials $p_l^{(L,n)}$, orthogonal on a sufficiently big but finite interval $[-L, L]$ with respect to the same weight (2.17). This will simplify our analysis and is justified by the following

Lemma 3.1 *Let $\{p_l^{(L,n)}\}_{l=0}^{\infty}$ is the system of polynomials orthogonal on the interval $[-L, L]$ with respect to the weight (2.17):*

$$\int_L^L p_l^{(L,n)}(\lambda) p_m^{(L,n)}(\lambda) e^{-nV(\lambda)} = \delta_{l,m}.$$

Denote by $\psi_l^{(L,n)}$, $K_n^{(L)}$, and $J_l^{(L,n)}(n)$ the quantities defined in (2.16), (2.15), and (2.19) for the system $\{p_l^{(L,n)}(\lambda)\}_{l=0}^{\infty}$. Assume, that $V(\lambda)$ satisfies conditions (1.3) and (1.4). Then there exist some absolute constants L and L_1 such that for any $0 \leq l \leq (1 + \epsilon/4)n$

$$\max_{|\lambda| \leq L} \left| \psi_l(\lambda) - \psi_l^{(L)}(\lambda) \right| \leq C e^{-nL_1}, \quad (\lambda \in [-L, L]). \tag{3.4}$$

$$|J_l^{(L,n)} - J_l^{(n)}| \leq C e^{-nL_1} \tag{3.5}$$

$$\max_{|\lambda|,|\mu| \leq L} |K_n(\lambda, \mu) - K_n^{(L)}(\lambda, \mu)| \leq C e^{-nL_1} \tag{3.6}$$

The lemma allows us to substitute $R_l^{(n)}$, $E_n(\Delta_n)$ and $J_l^{(n)}$ in (2.23), (2.28) and (3.1) by the respective quantities, constructed from the polynomials $\{p_l^{(L,n)}\}_{l=0}^{\infty}$. We will assume from now on that this replacement is made and we will omit the super index (L) to simplify notations.

Now we will prove the first assertion of the theorem, relations (2.23) - (2.27). Since any permutation of l objects can be represented as the product of the cyclic permutations, each term of the determinant in (2.13) is the product of the expressions $K_n(\lambda_1, \lambda_2) \ldots K_n(\lambda_{m-2}, \lambda_{m-1}) K_n(\lambda_{m-1}, \lambda_1)$, in which the arguments are in the cyclic order and do not appear in any other cyclic expression of the product. Hence, it suffices to prove the weak limit

$$\lim_{n \to \infty} (\gamma n^{2l/3})^{-l} K_n \left(a + t_1/\gamma n^{2/3}, a + t_2/\gamma n^{2/3} \right) \ldots K_n \left(a + t_m/\gamma n^{2/3}, a + t_1/\gamma n^{2/3} \right) = \mathcal{K}(t_1, t_2) \ldots \mathcal{K}(t_m, t_1) \tag{3.7}$$

for any $m \geq 1$. We will confine ourselves to the case $m = 2$, containing all important ingredients of the general case.

We set $z_1 = a + n^{-2/3}\zeta_1$, $z_2 = a + n^{-2/3}\zeta_2$ and introduce the function

$$F_n(\zeta_1, \zeta_2) = n^{-4/3} \int\int \Im \frac{1}{\lambda_1 - z_1} \Im \frac{1}{(\lambda_2 - z_2)} K_n^2(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \tag{3.8}$$

for $|\Im \zeta_{1,2}| \geq \varepsilon_0 > 0$, i.e., the two-dimensional Poisson integral of the function $K_n^2(\lambda_1, \lambda_2)$. According to (2.16) - (2.19) the functions $\{\psi_l^{(n)}\}_{l=0}^{\infty}$ are the generalized eigenfunctions of the selfadjoint operator,

defined in the space $l^2(\mathbf{Z}_+)$ by the matrix $J^{(n)}$ of (2.21). Denoting the operator again $J^{(n)}$, we introduce its resolvent

$$R^{(n)}(z) = (J^{(n)} - zI)^{-1}, \tag{3.9}$$

and the matrix $\{R_{j,k}^{(n)}(z)\}_{j,k=0}^{\infty}$ of the resolvent in the canonical basis of $l^2(\mathbf{Z}_+)$. Then the spectral theorem yields the representation

$$R_{j,k}^{(n)}(z) = \int \frac{\psi_j^{(n)}(\lambda)\psi_k^{(n)}(\lambda)}{\lambda - z}d\lambda, \quad \Im z \neq 0. \tag{3.10}$$

By using (2.15) and this representation we obtain that the function F_n of (3.8) can be written as follows:

$$F_n(\zeta_1, \zeta_2) = n^{-4/3} \sum_{j,k=1}^{n} \Im R_{n-j,n-k}^{(n)}(a + n^{-2/3}\zeta_1)\Im R_{n-j,n-k}^{(n)}(a + n^{-2/3}\zeta_2). \tag{3.11}$$

Set

$$M = [n^{3/5}], \tag{3.12}$$

Lemma 3.2 *Let $R^{(n)}(z)$ be the resolvent (3.9) of $J^{(n)}$. Then for any $z = a + n^{-2/3}\zeta$, $|\Im\zeta| > e^{-C\sqrt{n}}$, $\Re\zeta \geq -C$ we have*

$$n^{-4/3} \sum_{j=M+1}^{n} \sum_{k=0}^{\infty} |R_{n-j,k}^{(n)}(z)|^2 \leq C\frac{n^{5/3}}{M^3} = Cn^{-2/15}. \tag{3.13}$$

The proof of the lemma will be given in the next section.

Consider the operator

$$A = \frac{a}{2}\frac{d^2}{dx^2} - 2cx \tag{3.14}$$

defined on the whole real line. Denote by $R(\zeta)$ the resolvent $(A - \zeta I)^{-1}$ of A for $\Im\zeta \neq 0$, and by $R_\zeta(x,y)$ the kernel of $R(\zeta)$. We will need the following

Proposition 3.3 *The kernel $R_\zeta(x,y)$ possesses the properties:*

(i)

$$\frac{a}{2}\frac{\partial^2}{\partial x^2}R_\zeta(x,y) - 2cxR_\zeta(x,y) = \zeta R_\zeta(x,y) + \delta(x-y), \tag{3.15}$$

(ii)

$$R_\zeta(x,y) = \frac{2\pi}{\kappa a}\begin{cases} \psi_-(x,\zeta)\psi_+(y,\zeta) & x \leq y, \\ \psi_+(x,\zeta)\psi_-(y,\zeta) & x \geq y, \end{cases} \tag{3.16}$$

where

$$\psi_+(x,\zeta) = \text{Ai}(X), \quad \psi_-(x,\zeta) = \text{Ci}(X), \tag{3.17}$$

$$\text{Ci}(X) = i\text{Ai}(X) - \text{Bi}(X) \tag{3.18}$$

$X = \kappa x + \gamma\zeta$, $\kappa = (4ca^{-1})^{1/3}$, *and* $\text{Ai}(z)$ *and* $\text{Bi}(z)$ *are the standard Airy functions (see [1]).*

(iii) The functions $\psi_\pm$ are entire in x and ζ and have the following asymptotic behavior in x for $\Im\zeta = \varepsilon > 0$

$$|\psi_+(x,\zeta)| = \frac{1}{\pi^{1/2}|X|^{1/4}}(1 + O(|x|^{-3/2}))\begin{cases} \exp\{-\frac{2}{3}|\Re X|^{3/2}\}, & x \to +\infty \\ \exp\{\gamma\varepsilon|\Re X|^{1/2}\}, & x \to -\infty \end{cases} \tag{3.19}$$

$$|\psi_-(x,\zeta)| = \frac{1}{2\pi^{1/2}|X|^{1/4}}(1 + O(|x|^{-3/2}))\begin{cases} \exp\{\frac{2}{3}|\Re X|^{3/2}\}, & x \to +\infty \\ \exp\{-\gamma\varepsilon|\Re X|^{1/2}\}, & x \to -\infty \end{cases} \tag{3.20}$$

(iv) if $I(x,y) = \Im R_\zeta(x,y)$, then

$$|I(x,y)|^2 \leq I(x,x)I(y,y), \text{ and } \int_{-\infty}^{0} I(x,x)dx < \infty. \tag{3.21}$$

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The proposition will be proved in the next section.

Introduce the double infinite matrix

$$R^*_{l_1,l_2}(z) = n^{1/3} R_\zeta \left(\frac{n-l_1}{n^{1/3}}, \frac{n-l_2}{n^{1/3}} \right), \quad z = a + n^{-2/3}\zeta, \tag{3.22}$$

and the semi infinite matrix

$$D = \{d_{l_1,l_2}\}_{l_1,l_2=0}^{\infty}, \ D = (J^{(n)} - zI)R^*(z) - I. \tag{3.23}$$

Then we have

$$R^{(n)}(z) = R^*(z) - R^{(n)}D(z). \tag{3.24}$$

We introduce also the $(4M+1) \times (4M+1)$ matrix $D^{(M)}$, assuming that n is big enough and setting

$$D^{(M)} = \{D^{(M)}_{l_1,l_2}\}, \ D^{(M)}_{n-j,n-k} = \begin{cases} d_{n-j,n-k}, & |j|,|k| \leq 2M, \\ 0, & \text{otherwise.} \end{cases} \tag{3.25}$$

We will use the following lemmas, that will also be proved in the next section.

Lemma 3.4 *Under conditions of Theorem 2.2 for any $z = a + n^{-2/3}\zeta$ with $|\Re\zeta| \leq C$ and $|\Im\zeta| \geq \varepsilon = O(n^{-\alpha_1})$, $0 \leq \alpha_1 \leq 1/11$, $\alpha_2 \geq 0$ we have*

$$n^{-4/3} \sum_{j=1}^{M} \sum_{|k| \leq M} |R^{(n)}_{n-j,n-k}(z)|^2 \leq 2n^{-4/3} \sum_{j=1}^{M} \sum_{|k| \leq 2M} |R^*_{n-j,n-k}(z)|^2 + CM/n^{5/2} + O(n^{-1}), \tag{3.26}$$

$$n^{-4/3} \sum_{j=1}^{M} \sum_{|k| \leq M} |(R^{(n)}D)_{n-j,n-k}(z)|^2 \leq C\varepsilon^{-1}||D^{(M)}||^2. \tag{3.27}$$

Lemma 3.5 *If $|\Im\zeta| \geq \varepsilon > 0$, then the norm of the matrix $D^{(M)}$ admits the bound*

$$||D^{(M)}|| \leq C \left((M/n)^{1/2} + \varepsilon^{-1}M^2/n^{4/3} \right). \tag{3.28}$$

On the basis of (3.13) and (3.24), we get for (3.11)

$$F(\zeta_1,\zeta_2) = n^{-4/3} \sum_{j,k=1}^{M} \Im R^{(n)}_{n-j,n-k}(z_1) \Im R^{(n)}_{n-j,n-k}(z_2) + O(n^{-2/15})$$

$$= n^{-4/3} \sum_{j,k=1}^{M} \Im R^*_{n-j,n-k}(z_1) \Im R^*_{n-j,n-k}(z_2) + \delta_n(\zeta_1,\zeta_2) + O(n^{-2/15}), \tag{3.29}$$

where

$$|\delta_n(\zeta_1,\zeta_2)| = \left| n^{-4/3} \sum_{j,k=1}^{M} \left[\Im(R^{(n)}D)_{n-j,n-k}(z_1) \Im R^*_{n-j,n-k}(z_2) + \right. \right.$$

$$\left. \Im(R^{(n)}D)_{n-j,n-k}(z_2) \Im R^*_{n-j,n-k}(z_1) + \Im(R^{(n)}D)_{n-j,n-k}(z_1) \Im(R^{(n)}D)_{n-j,n-k}(z_2) \right|$$

$$\leq \left[n^{-4/3} \sum_{j,k=1}^{M} |(R^{(n)}D)_{n-j,n-k}(z_1)|^2 \right]^{1/2} \left[n^{-4/3} \sum_{j,k=1}^{M} |R^*_{n-j,n-k}(z_2)|^2 \right]^{1/2}$$

$$+ \left[n^{-4/3} \sum_{j,k=1}^{M} |(R^{(n)}D)_{n-j,n-k}(z_2)|^2 \right]^{1/2} \left[n^{-4/3} \sum_{j,k=1}^{M} |R^*_{n-j,n-k}(z_1)|^2 \right]^{1/2}$$

$$+ \left[n^{-4/3} \sum_{j,k=1}^{M} |(R^{(n)}D)_{n-j,n-k}(z_1)|^2 \right]^{1/2} \left[n^{-4/3} \sum_{j,k=1}^{M} |(R^{(n)}D)_{n-j,n-k}(z_2)|^2 \right]^{1/2}$$

Since we have to prove the convergence of F_n for $\zeta_{1,2}$ such that $|\zeta_{1,2}| \leq C$ and $|\Im\zeta_{1,2}| \geq \varepsilon_0 > 0$, we can take an n-independent $\varepsilon = \varepsilon_0$ in Lemmas 3.4, and 3.5. Then, by using the Euler-MacLaurin

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summation formula [1], it can be shown that the sum in the l.h.s. of (3.26) is $O(1)$ as $n \to \infty$. This leads to the bound

$$|\delta_n(\zeta_1, \zeta_2)| \leq C\, n^{-2/15}.$$

The bound, Proposition 3.3, (3.29), and (3.22) imply that for any $\zeta_{1,2}$ such that $|\zeta_{1,2}| \leq C < \infty$ and $|\Im\zeta_{1,2}| \geq \varepsilon_0 > 0$, we have

$$\lim_{n\to\infty} F_n(\zeta_1, \zeta_2) = \int_0^\infty \int_0^\infty \Im R_{\zeta_1}(x_1, x_2)\Im R_{\zeta_2}(x_1, x_2)dx_1 dx_2. \tag{3.30}$$

According to (3.16) the function $(2/\kappa a)^{1/2} Ai(\kappa x + \gamma\xi)$ is the generalized eigenfunction of the self-adjoint operator (3.14), corresponding to the generalized eigenvalue $\xi \in \mathbf{R}$. This fact and the spectral theorem for the operator yield the integral representation

$$\Im R_\zeta(x, y) = \frac{2}{\kappa a} \int Ai(\kappa x + \gamma\xi)Ai(\kappa y + \gamma\xi)\Im\frac{1}{\xi - \zeta}d\xi. \tag{3.31}$$

The formula and Proposition 3.3 allow us to rewrite the r.h.s. of (3.30) as

$$\gamma^2 \int_0^\infty \int_0^\infty \Im\frac{1}{\xi_1 - \zeta_1}\Im\frac{1}{\xi_2 - \zeta_2}d\xi_1 d\xi_2$$
$$\times \int_0^\infty \int_0^\infty dx_1 dx_2 Ai(x_1 + \gamma\xi_1)Ai(x_1 + \gamma\xi_2)Ai(x_2 + \gamma\xi_1)Ai(x_2 + \gamma\xi_2).$$

To finish the proof we need the following lemma, proved in the next section.

Lemma 3.6 *Under conditions of Theorem 2.2 for any fixed $L_0 > -\infty$*

$$n \int_{\mathbf{R}\setminus\sigma(n^{-2/3}L_0)} \rho_n(\lambda)d\lambda \leq C \int_{L_0}^\infty dt \int_0^\infty dx Ai^2(\kappa x + \gamma t) + o(1), \quad n \to \infty, \tag{3.32}$$

where $\sigma(\varepsilon)$ for $\varepsilon > 0$ is the ε-neighborhood of σ and for $\varepsilon < 0$ $\sigma(\varepsilon)$ denotes the part of σ, whose distance from the boundary of σ exceeds $|\varepsilon|$.

Since the l.h.s. of (3.30) is bounded from above for $|\Im\zeta| > 0$, we conclude that the sequence of measures in $\mathbf{R}^2$, defined by the densities $n^{-4/3}K_n^2(a + n^{-2/3}\xi_1, a + n^{-2/3}\xi_2)$ (cf (2.29)), is weakly compact. Besides, Proposition 3.6 and the inequality

$$K_n^2(\lambda, \mu) \leq K_n(\lambda, \lambda)K_n(\mu, \mu) = n^2\rho_n(\lambda)\rho_n(\mu) \tag{3.33}$$

imply that the contributions of neighborhoods of $\pm\infty$ with respect to the both variables in (3.8) is negligible uniformly in n. Since the Poisson integral determines uniquely the corresponding measure, we deduce from the above and formulas (2.32), (3.11), and (3.30) the tight convergence of $n^{-4/3}K_n^2(a + n^{-2/3}\xi_1, a + n^{-2/3}\xi_2)d\xi_1 d\xi_2$ to $\gamma^2\mathcal{K}^2(\gamma\xi_1, \gamma\xi_2)d\xi_1 d\xi_2$. Changing variables $\xi_{1,2}$ to $t_{1,2}/\gamma$, we obtain (2.23) - (2.24) for $l = 2$.

This finishes the proof of the first assertion of the theorem, i.e. the relations (2.23)–(2.26), for the single interval support (2.2) of the Density of States of the ensemble. Let us prove now the second assertion of the theorem, formula (2.28). We note first that if $\Delta_n = a + \Delta/\gamma n^{2/3}$, then we have according to (2.14)

$$E_n(\Delta_n) = 1 + \sum_{l=1}^n \frac{(-1)^l}{l!} \int_\Delta dt_1 \ldots dt_l (\gamma n^{2/3})^{-l} \det\left\{K_n(a + t_j/\gamma n^{2/3}, a + t_k/\gamma n^{2/3})\right\}_{j,k=1}^l, \tag{3.34}$$

Recall now the Hadamard inequality, according to which we have for any $l \times l$ matrix $A = \{A_{jk}\}_{j,k=1}^l$:

$$|\det A| \leq \prod_{j=1}^l \left(\sum_{k=1}^n |A_{jk}|^2\right)^{1/2}.$$

If the matrix is positive definite, the inequality can be modified as follows

$$\det A \leq \prod_{j=1}^l A_{jj}. \tag{3.35}$$

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The last inequality and Proposition 3.6 allow us to make the limit $n \to \infty$ in the r.h.s. of (2.28), hence to prove (2.28) for any set $\Delta \subset \mathbf{R}$, finite or bounded from the left.

This finishes the proof of Theorem 2.2 for the case (2.2) of a single interval support of the Density of States of the ensemble and the right hand endpoint a of the support. The case of the left hand endpoint $-a$ can be treated analogously by setting $z = -a - n^{-2/3}\zeta$ and by using

$$-n^{1/3}(-1)^{l_1+l_2}(A-\zeta)^{-1}\left(\frac{n-l_1}{n^{1/3}}, \frac{n-l_1}{n^{1/3}}\right)$$

as the matrix R^*.

To prove the theorem for the case (2.1) of a two-interval support we note that now we have the following asymptotic relation [2] (cf (3.1)):

$$J^{(n)}_{n+k} = \frac{1}{2}(b - (-1)^k a) + \frac{k}{n(b^2-a^2)}\left(\frac{1}{P(b)} - \frac{(-1)^k}{P(a)}\right) + r^{(n)}_k \tag{3.36}$$

instead of (2.21), where $r^{(n)}_k$ again satisfies (3.2). As a result, in the case the endpoint b we should consider instead (3.14) the operator

$$A^{(b)} = \frac{b^2-a^2}{2b}\frac{d^2}{dx^2} - \frac{2}{(b^2-a^2)P(b)}x$$

with the resolvent $R^{(b)}(\zeta) = (A^{(b)} - \zeta I)^{-1}$ whose kernel is $R^{(b)}_\zeta(x,y)$. Then we consider the matrix $R^{(*,b)}(\zeta)$ of the form

$$R^{(*,b)}_{n-j,n-k}(\zeta) = n^{1/3}R^{(b)}_\zeta\left(\frac{j}{n^{1/3}} + \frac{(-1)^j a}{2n^{1/3}b}, \frac{k}{n^{1/3}} + \frac{(-1)^k a}{2n^{1/3}b}\right). \tag{3.37}$$

The respective operator for the endpoint a is

$$A^{(a)} = \frac{b^2-a^2}{2a}\frac{d^2}{dx^2} + \frac{2}{(b^2-a^2)P(a)}x$$

with the resolvent $R^{(a)}(\zeta)$ and

$$R^{(*,a)}_{n-j,n-k}(\zeta) = n^{1/3}(-1)^{[\frac{k+1}{2}]}(-1)^{[\frac{j+1}{2}]}R^{(a)}_\zeta\left(\frac{j}{n^{1/3}} + \frac{(-1)^j b}{2n^{1/3}a}, \frac{k}{n^{1/3}} + \frac{(-1)^k b}{2n^{1/3}a}\right) \tag{3.38}$$

where $R^{(a)}_\zeta(x,y)$ be the kernel of $R^{(a)}(\zeta)$.

Theorem 2.2 is proved.

4 Auxiliary results

Proof of Lemma 3.1 We prove first (3.4). For $\overline{\mu} = (\mu_1, \ldots, \mu_k)$ we set:

$$\Pi_k(\overline{\mu}) = \prod_{1 \le i < j \le k}(\mu_i - \mu_j)^2 \prod_{i=1}^k e^{-nV(\mu_i)}, \quad \Pi_k^*(\lambda, \overline{\mu}) = e^{-nV(\lambda)/2}\prod_{i=1}^k(\lambda - \mu_i)$$

$$Q^{(n)}_k = \int d\overline{\mu}\Pi_k(\overline{\mu}), \quad Q^{(L,n)}_k = \int_{[-L,L]^k} d\overline{\mu}\Pi_k(\overline{\mu}),$$

In particular, $Q^{(n)}_n$ is Q_n of (2.10). Then, according to [21] (see formulas (2.2.10) - (2.2.11)), we have

$$\psi^{(n)}_k(\lambda) = \frac{\gamma^{(n)}_k}{Q^{(n)}_k}\int \Pi_k^*(\lambda, \overline{\mu})\Pi_k(\overline{\mu})d\overline{\mu}, \quad \psi^{(L,n)}_k(\lambda) = \frac{\gamma^{(L,n)}_k}{Q^{(L,n)}_k}\int_{[-L,L]^k} \Pi_k^*(\lambda, \overline{\mu})\Pi_k(\overline{\mu})d\overline{\mu}, \tag{4.1}$$

where $\gamma^{(n)}_k$ and $\gamma^{(L,n)}_k$ are the coefficients in front of λ^k in $p^{(n)}_k(\lambda)$ and $p^{(L,n)}_k(\lambda)$. They have the form [21]

$$(\gamma^{(n)}_k)^2 = \frac{Q^{(n)}_k(k+1)}{Q^{(n)}_{k+1}}, \quad (\gamma^{(L,n)}_k)^2 = \frac{Q^{(L,n)}_k(k+1)}{Q^{(L,n)}_{k+1}}. \tag{4.2}$$

We prove first that for some $L, \tilde{L}_1$ uniformly in $k \le (1 + \epsilon/4)n$ we have

$$\left| \frac{Q_k^{(n)}}{Q_k^{(L,n)}} - 1 \right| \le Ce^{-nL_1}, \quad \left| \frac{\gamma_k^{(n)}}{\gamma_k^{(L,n)}} - 1 \right| \le Ce^{-nL_1}. \tag{4.3}$$

The first relation here follows from the result of [7], Lemma 1, which states that for any function V, satisfying conditions (1.3) with some $\tilde{\epsilon} > 0$ and (1.4) there exist absolute constants $L > 1, \tilde{L}_1$ such that for any $k \le n$

$$\begin{aligned}
\left| \frac{\tilde{Q}_k^{(n)}}{\tilde{Q}_k^{(L,n)}} - 1 \right| &\le Ce^{-n\tilde{L}_1}, \\
(1 - \chi_L(\mu_1))\rho_k^{(n)}(\mu_1) &\le C \exp\{-n\tilde{L}_1 \log |\mu_1|\}, \\
(1 - \chi_L(\mu_1))\rho_k^{(n)}(\mu_1, \mu_2) &\le C \exp\{-n\tilde{L}_1 \log |\mu_1|\},
\end{aligned} \tag{4.4}$$

where χ_L is the characteristic function of the interval $[-L, L]$ and

$$\rho_k^{(n)}(\mu_1) = (Q_k^{(n)})^{-1} \int \Pi_k(\overline{\mu}) d\mu_2, \ldots d\mu_k, \quad \rho_k^{(n)}(\mu_1, \mu_2) = (Q_k^{(n)})^{-1} \int \Pi_k(\overline{\mu}) d\mu_3, \ldots d\mu_k, \tag{4.5}$$

are the first and the second marginals of the probability density $\Pi_k(\overline{\mu})/Q_k^{(n)}$ of k variables $\overline{\mu} = (\mu_1, \ldots, \mu_k)$, corresponding to the potential $\tilde{V} = nk^{-1}V$ (cf (2.10), (2.11)).

Thus, if we chose constants $L, \tilde{L}_1$ for $\tilde{V}(\lambda) = (1 + \epsilon/4)V(\lambda)$, $\tilde{\epsilon} = \epsilon/4(1 + \epsilon/4)$, we obtain the first bound of (4.3). The second bound follows from the first in view of the relations (4.2).

Now if we denote by $\Delta_k(\lambda)$ the r.h.s. of (3.4), then, using (4.2), (4.3), and the second line of (4.4), we get

$$\Delta_{k+1}(\lambda) \le Ce^{-n\tilde{L}_1} + \left| \frac{\gamma_k^{(n)}}{Q_{k-1}^{(n)}} \left(\int d\overline{\mu} - \int_{[-L,L]^n} d\overline{\mu} \right) \Pi_k^*(\lambda, \overline{\mu}) \Pi_k(\overline{\mu}) \right| \tag{4.6}$$

Denoting $\Delta_k'(\lambda)$ the second term in the r.h.s. of (4.6), we obtain

$$\begin{aligned}
\Delta_k'(\lambda) &\le n\frac{\gamma_k^{(n)}}{Q_k^{(n)}} \int (1 - \chi_L(\mu_1))\Pi_k^*(\lambda, \overline{\mu})\Pi_k(\overline{\mu}) d\overline{\mu} \\
&\le n\sqrt{k+1} \left[(Q_k^{(n)})^{-1} \int d\overline{\mu}(1 - \chi_L(\mu_1))\Pi_k(\overline{\mu}) d\overline{\mu} \right]^{1/2} \\
&\quad \left[(Q_{k+1}^{(n)})^{-1} \int d\overline{\mu}(1 - \chi_L(\mu_1))(\Pi_k^*(\lambda, \overline{\mu}))^2 \Pi_k(\overline{\mu}) d\overline{\mu} \right]^{1/2} \\
&\le nL^{1/2}\sqrt{k+1} \left| \int (1 - \chi_L(\mu_1))\rho_k^{(n)}(\mu_1) d\mu_1 \right|^{1/2} \left| \int (1 - \chi_L(\mu_1))\rho_{k+1}^{(n)}(\lambda, \mu_1) d\mu_1 \right|^{1/2}
\end{aligned} \tag{4.7}$$

Here we have used (4.2) and (4.5). According to (4.4) the integrals in the r.h.s. of (4.7) are $O(n^{-1}e^{-n\tilde{L}_1|\log L|})$. Thus, taking $L_1 = \tilde{L}_1 \log L/2$, we get (3.4) if n is large enough.

It follows from (2.19) that

$$J_k^{(n)} = \gamma_k^{(n)}/\gamma_{k+1}^{(n)}.$$

This and (4.3) imply (3.5). Another way to prove (3.5) is to apply (3.4) and the second line of (4.4) to formula (3.3).

The proof of (3.6) follows from (2.15) and (3.4)

Proof of Proposition 3.3. By general principles (see e.g.[4]) the kernel $R_\zeta(x, y)$ of the resolvent of differential operator (3.14) has the form

$$R_\zeta(x, y) = \frac{1}{W} \begin{cases} \psi_-(x)\psi_+(y) & x \le y, \\ \psi_-(y)\psi_+(x) & x \ge y, \end{cases}$$

where $f_\pm(x, \zeta)$ are solutions of the differential equations

$$\frac{a}{2}\psi''(x) - (2cx + \zeta)\psi(x) = 0,$$

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that are square integrable at $\pm\infty$, and W is fixed be the condition

$$\frac{\partial}{\partial x}R_z(x+0,x) - \frac{\partial}{\partial x}R_z(x-0,x) = \frac{2}{a}.$$

According to formulas (4.8), we can choose $\psi_+(x) = \text{Ai}(\kappa x + \gamma\zeta)$ and $\psi_-(x) = \text{Ci}(\kappa x + \gamma\zeta) = i\text{Ai}(\kappa x + \gamma\zeta) + \text{Bi}(\kappa x + \gamma\zeta)$. This and the identity (see [1]),

$$(\text{Ai})'(z)\text{Ci}(z) - \text{Ai}(z)(\text{Ci})'(z) = \pi^{-1}.$$

lead to (3.16) - (3.18).

Relations (3.19), (3.20) follow from the asymptotic representations (see [1])

$$
\begin{aligned}
\text{Ai}(z) &= \pi^{-1/2}z^{-1/4}e^{-\frac{2}{3}z^{3/2}}\left(1 + O(z^{-3/2})\right), & |\arg z| < \pi, \\
\text{Ai}(-z) &= \pi^{-1/2}z^{-1/4}\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)\left(1 + O(z^{-3/2})\right), & |\arg z| < \frac{2}{3}\pi, \\
\text{Ci}(z) &= \pi^{-1/2}z^{-1/4}e^{\frac{2}{3}z^{3/2}}\left(1 + O(z^{-3/2})\right), & |\arg z| < \frac{\pi}{3}, \\
\text{Ci}(-z) &= \pi^{-1/2}z^{-1/4}\exp\left\{i\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)\right\}\left(1 + O(z^{-3/2})\right), & |\arg z| < \frac{2}{3}\pi.
\end{aligned}
\tag{4.8}
$$

The leading terms of the asymptotic formulas for the derivatives of Ai and Ci can be obtained as the leading terms of the formal derivatives at the above formulas. This and (3.16) lead to assertion (iii) of the proposition.

Assertion (iv) follows from (ii) - (iii) and (4.8).

Proof of Lemma 3.2 It is easy to see, that the l.h.s. of (3.13) is bounded above by the expression

$$\sum_{k=0}^{n-M}\sum_{m=0}^{\infty}|R_{k,m}^{(n)}(z)|^2 = \sum_{k=1}^{n-M}(R^{(n)}(z)R^{(n)}(\overline{z}))_{kk} = (n-M)\int\frac{\rho_{n-M}^{(n)}(\lambda)d\lambda}{|x-z|^2}, \tag{4.9}$$

where $\rho_{n-M}^{(n)}(\lambda)$ is the first marginal density of the probability density (cf (2.10))

$$p_{n-M}^{(n)}(\lambda_1,...\lambda_{n-M}) = \left(Q_{M-n}^{(n)}\right)^{-1}\prod_{1\le j<k\le n-M}(\lambda_j-\lambda_k)^2\exp\{-(n-M)\sum_{j=1}^{n-M}\frac{1}{1-M/n}\cdot V(\lambda_j)\} \tag{4.10}$$

This suggest the introduction of the functional

$$\mathcal{E}_\delta[m] = \frac{1}{1-\delta}\int V(\lambda)m(d\lambda) - \int\int\log|\lambda-\mu|m(d\lambda)m(d\mu) \tag{4.11}$$

with $\delta \in (0,1)$ (the functional (1.6)) corresponds to the case $\delta = 0$ of the above functional. According to the results of [7] (see also [15]), if V satisfies (1.3), and V' is a locally Hölder function, then the unique minimizer of the functional (4.11) is equal to the limit $\rho^{(\delta)}$ of the first marginal density $\rho_{n-M}^{(n)}$ of the distribution (4.10) as $n \to \infty$, $M \to \infty$, $M/n \to \delta$, the limit is in a certain "energy" norm, determined by the second term of (4.11). It is easy to find (see also [22, 16]), that the support σ_δ of the density $\rho^{(\delta)}$ lies strictly inside of the support σ of the DOS (i.e., the density of the limit of the first marginal density of the distribution (2.10), corresponding to $\delta = 0$ in (4.11)). Moreover, the endpoint a_δ of σ_δ and the endpoint a of σ are related as follows

$$\lim_{\delta\to 0}(a - a_\delta)\delta^{-1} = K > 0, \tag{4.12}$$

where K can be written via derivatives of the function $u(x)$, defined in (2.3) (this relation can also be deduced from (4.18) in [2]).

Now we need the following proposition, proven in [2] (see the last inequality in the proof of Proposition 2 and the text before the inequality).

Proposition 4.1 *Consider a unitary invariant ensemble of the form (1.1)–(1.2) and assume that $V(\lambda)$ possesses two bounded derivatives in some neighborhood of the support σ of the Density of States ρ, that satisfies condition C2. Denote by $\sigma(\varepsilon)$ the ε-neighborhood of the spectrum σ. Then there exist C_1, C_2*

$$\int_{\mathbf{R}\backslash\sigma(C_1 n^{-1/2}\log n)}\rho_n(\lambda)d\lambda \le e^{-C_2 n^{1/2}\log n}. \tag{4.13}$$

According to Proposition 4.1, we have uniformly in $0 \leq M/n \leq \delta_0 < 1$

$$\int_{a_\delta + C_1 n^{-1/2} \log n} \rho^{(n)}_{n-M}(\lambda) d\lambda \leq \exp\{-C_2 n^{1/2} \log n\}, \tag{4.14}$$

where C_1 and C_2 may depend on δ_0.

Set $\varepsilon(n) = KM/2n$. Then $\varepsilon(n) >> C_1 n^{-1/2} \log n$, and so $a - \varepsilon(n) > a_\delta + C_1 n^{-1/2} \log n$ for sufficiently big n and $M >> n^{1/2} \log n$. Thus, we can write:

$$
\begin{aligned}
\int \frac{\rho^{(n)}_{n-M}(\lambda) d\lambda}{|a + n^{-2/3}\zeta - \lambda|^2} &= \left[\int_{\lambda > a - \varepsilon(n)} + \int_{\lambda \leq a - \varepsilon(n)} \right] \frac{\rho^{(n)}_{n-M}(\lambda) d\lambda}{(a + n^{-2/3}\zeta - \lambda)^2} \\
&\leq \frac{n^{4/3}}{|\Im\zeta|^2} \exp\{-C_0 n^{1/2} \log n\} + \int_{\lambda \leq a - \varepsilon(n)} \frac{\rho^{(n)}_{n-M}(\lambda) d\lambda}{|\lambda - a - n^{-2/3}\Re\zeta|^2 + n^{-4/3}|\Im\zeta|^2} \\
&\leq O(e^{-C \cdot \sqrt{n}}) + C \int \frac{\rho^{(n)}_{n-M}(\lambda) d\lambda}{|\lambda - a|^2 + (M/n)^2} \leq O(e^{-C \cdot \sqrt{n}}) + C |\Im g^{(n)}_{M-n}(a + iM/n)| n/M,
\end{aligned}
\tag{4.15}
$$

where $g^{(n)}_{n-M}(z)$ is the Stieltjes transform of the measure $\rho^{(n)}_{n-M}(\lambda) d\lambda$ (see (2.7)). According to the results of [20], Eqs.(2.5) - (2.6), if $z \in \mathbf{D}$, then $g^{(n)}_{n-M}(z)$ verifies the relation

$$(g^{(n)}_{n-M}(z))^2 - \frac{1}{1 - M/n} V'(z) g^{(n)}_{n-M}(z) + \mathcal{Q}^{(M/n)}_n(z) = O(\frac{1}{n^2 |\Im z|^4}), \quad n \to \infty, \tag{4.16}$$

where (cf (2.9))

$$\mathcal{Q}^{(M/n)}_n(z) = \frac{1}{1 - M/n} \int \frac{V'(z) - V'(x)}{z - \lambda} \rho^{(n)}_{n-M}(\lambda) d\lambda. \tag{4.17}$$

On the other hand, if we consider $g^{(\delta)}(z)$, the Stieltjes transform of the measure $\rho^{(\delta)}(\lambda) d\lambda$, minimizing (4.11), then for $z \in \mathbf{D}$ $g^{(\delta)}(z)$ is a solution of the quadratic equation (see (2.8)–(2.9)

$$(g^{(\delta)}(z))^2 - \frac{1}{1 - \delta} \mathcal{V}'(z) g^{(\delta)}(z) + \mathcal{Q}^{(\delta)}(z) = 0 \tag{4.18}$$

in the class of analytic functions, such that $\Im z \cdot \Im g > 0$, and $\mathcal{Q}^{(\delta)}(z)$ is defined by the formula (4.17) with $\rho^{(\delta)}$ instead of $\rho^{(n)}_{n-M}$ and δ instead of M/n. Now (4.16)–(4.18) and the fact that the analytic function $(\mathcal{V}'(z)(1 - \delta)^{-1})^2 - 4\mathcal{Q}^{(\delta)}(z)$ is zero at $z = a_\delta$ imply

$$|\Im g^{(n)}_{n-M}(z)| \leq C(|\Im z| + |z - a_\delta|^{1/2} + |\mathcal{Q}^{(M/n)}_n(z) - \mathcal{Q}^{(M/n)}(z)|^{1/2} + n^{-1}|\Im z|^{-2}).$$

Besides, since $(\mathcal{V}'(z) - V'(\lambda))/(z - \lambda)$ has bounded derivative with respect to λ (recall that $\mathcal{V}$ is analytic in a certain neighborhood of the support) we can use results of paper [7], according to which

$$|\mathcal{Q}^{(M/n)}_n(z) - \mathcal{Q}^{(M/n)}(z)| \leq C \cdot n^{-1/2} \log^{1/2} n.$$

The last two inequalities and (4.12) yield

$$|\Im g^{(n)}_{n-M}(a + iM/n)| \leq C \cdot ((M/n)^{1/2} + n^{-1/4} \log^{1/4} n + n/M^2). \tag{4.19}$$

Now, using (4.15) and (4.19), we obtain from (4.15)

$$\int \frac{\rho^{(n)}_{n-M}(\lambda) d\lambda}{|a_j + n^{-2/3}\zeta - \lambda|^2} \leq C \cdot ((n/M)^{1/2} + n^2/M^3 + n^{3/4} \log^{1/4} n/M).$$

Combining this bound with (4.9), we obtain (3.13). Lemma 3.2 is proved.

Proof of Lemma 3.4 Take $\nu \in \mathbf{N}$ to provide the bound (see Lemma 3.5)

$$||D^{(M)}||^\nu \leq Cn^{-2}. \tag{4.20}$$

By (3.24) we have

$$R^{(n)} = R^* + \sum_{l=1}^{\nu-1} (-1)^l R^* D^l + (-1)^\nu R^{(n)} D^\nu, \tag{4.21}$$

We write for both $R = R^{(n)}$ and $R = R^*$ and $l = 1, ..., p$:

$$(RD^l)_{n-j,n-k} = \sum_{m_1,...,m_\ell=-2M}^{2M} R_{n-j,n-m_1} D_{n-m_1,n-m_1} \ldots D_{n-m_l,n-k} + d^{(l)}_{n-j,n-k} \qquad (4.22)$$

with

$$\begin{aligned} |d^{(l)}_{n-j,n-k}| &= \left| \sideset{}{'}\sum_l R_{n-j,n-m_1} D_{n-m_1,n-m_1} \ldots D_{n-m_l,n-k} \right| \\ &\leq \sum_{p=1}^{l} \sum_{|m|>2M} (|R||D|^{l-p})_{n-j,n-m} (|D|)^p_{n-m,n-k}, \end{aligned}$$

where $\sum'$ is the sum of the terms which contain at least one $|m_i| > 2M$ and $|R|_{l,l'} = |R_{l,l'}|$, $|D|_{l,l'} = |D_{l,l'}|$.

We observe that for $m \neq m'$ we have

$$|D|_{n-m,n-m'} \leq \max_i J_i^{(n)} (|R^*_{n-m-1,n-m'}(z)| + |R^*_{n-m+1,n-m'}(z)|) + 2a|R^*_{n-m,n-m'}(z)|), \qquad (4.23a)$$

where we also took into account that since the matrix $J^{(n)}$ corresponds now to the polynomials $p^{(L,n)}$, orthogonal on a finite interval $[-L, L]$ (see Lemma 3.1), the coefficients $J_i(n)$ are bounded uniformly in i and n. Hence, according to (3.19) and (3.20), for any $p = 1, \ldots, \nu$, $|k| \leq m$, $|m| > 2M$ we obtain for some positive α and β

$$\begin{aligned} (|D|^p)_{n-m,n-k} &\leq C\, n^\alpha \exp\{-C\varepsilon(Mn^{-1/3})^{1/2} - C\varepsilon((|m|-2M)n^{-1/3})^{1/2}\} \\ &\leq C\, e^{-C\varepsilon n^{2/15}} \exp\{-C\varepsilon((|m|-2M)n^{-1/3})^{1/2}\}, \end{aligned} \qquad (4.24)$$

and

$$|(|D|^p(|D|^\dagger)^p)_{n-m,n-m}| \leq C\, n^\alpha m^\beta, \quad |m| > 2M. \qquad (4.25)$$

We will also use the trivial bounds: $||R^{(n)}|| \leq n^{2/3}\varepsilon^{-1}$,

$$|(|R^{(n)}||D|^p)_{n-j,n-m}| \leq ||R^{(n)}|| \cdot |(|D|^p(|D|^\dagger)^p)_{n-m,n-m}|^{1/2},$$

$$|(R^*|D|^p)_{n-j,n-m}| \leq (|R^*||R^*|^\dagger)_{n-j,n-j}^{1/2} \cdot |(|D|^p(|D|^\dagger)^p)_{n-m,n-m}|^{1/2}.$$

The above bounds yield

$$|d^{(l)}_{n-j,n-k}| \leq Ce^{-C\varepsilon(Mn^{-1/3})^{1/2}},$$

and we have from (4.21) and (4.22)

$$\begin{aligned} R^{(n)}_{n-j,n-k} &= R^*_{n-j,n-k} + \sum_{l=1}^{\nu-1}(-1)^l(R^*(D^{(M)})^l)_{n-j,n-k} \\ &\quad + (-1)^\nu(R^{(n)}(D^{(M)})^\nu)_{n-j,n-k} + O(e^{-C\varepsilon(Mn^{-1/3})^{1/2}}). \end{aligned} \qquad (4.26)$$

Thus, we get for sufficiently large n

$$n^{-4/3}\sum_{j=1}^{M}\sum_{|k|\leq M} |R^{(n)}_{n-j,n-k}(z)|^2 = n^{-4/3}\sum_{j=1}^{M}\sum_{|k|\leq 2M} |R^*_{n-j,n-k}(z)|^2 + \delta_n(z), \qquad (4.27)$$

where

$$\begin{aligned} |\delta_n(z)| &\leq Cn^{-4/3}\sum_{\ell=1}^{\nu-1}\sum_{j=1}^{M}(R^*(D^{(M)})^l(D^{(M)\dagger})^l R^{*\dagger})_{n-j,n-j} \\ &\quad + C\sum_{j=1}^{M}(R^{(n)}(D^{(M)})^\nu(D^{(M)\dagger})^\nu R^{(n)\dagger})_{n-j,n-j} + o(n^{-1}) \\ &\leq 2C||D^{(M)}||^2 n^{-4/3}\sum_{j=1}^{M}\sum_{|k|\leq 2M} |R^*_{n-j,n-k}|^2 + C||D^{(M)}||^{2\nu}\sum_{j=1}^{M}(R^{(n)}R^{(n)\dagger})_{n-j,n-j} + o(n^{-1}) \\ &\leq 2C||D^{(M)}||^2 n^{-4/3}\sum_{j=1}^{M}\sum_{|k|\leq 2M} |R^*_{n-j,n-k}|^2 + C\varepsilon^{-2}||D^{(M)}||^{2\nu} Mn^{4/3} + o(n^{-1}) \end{aligned}$$

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Then, in view of (4.20), we obtain (3.26). Inequality (3.27) can be proved similarly.

Proof of Lemma 3.5. By the direct calculation we find from (3.1) for $j \neq k$:

$$((J^{(n)} - zI)R^*(z))_{n-j,n-k} = -aR^*_{n-j,n-k}(z) + \frac{a}{2}R^*_{n-j+1,n-k}(z) + \frac{a}{2}R^*_{n-j-1,n-k}(z)$$
$$-n^{-2/3}\zeta R^*_{n-j,n-k}(z) - c\frac{j}{n}R^*_{n-j+1,n-k}(z) - c\frac{j-1}{n}R^*_{n-j-1,n-k}(z)$$
$$+r^{(n)}_j R^*_{n-j+1,n-k}(z) + r^{(n)}_{j-1}R^*_{n-j-1,n-k}(z) + r^{(n)}_j R^*_{n-j,n-k}(z).$$

By using the Taylor formula of the forth order for the second and the third terms and the same formula of the second order for the fifth and the sixth terms, we rewrite the r.h.s of the last formula as

$$n^{-1/3}\left(-\zeta R_\zeta(x,y) + \frac{a}{2}\frac{\partial^2}{\partial x^2}R_\zeta(x,y) - 2cxR_\zeta(x,y)\right)\Bigg|_{x=\frac{j}{n^{1/3}},y=\frac{k}{n^{1/3}}} + d_{n-j,n-k}(z). \qquad (4.28)$$

The expression in the brackets of the last equality is equal to zero because of equation (3.15) for $x \neq y$. The remainder $d_{n-j,n-k}(z)$ is a linear combination of

$$\frac{\partial^\alpha}{\partial x^\alpha}R_\zeta(x,k/n^{1/3}), \quad \alpha = 0,2,4,$$

with $x = (j+\theta)/n^{2/3}$, where $|\theta| \leq 1$ can be different for different terms. The derivatives can be excluded by using equation (3.15) with corresponding θ. This leads to the bound for the remainder $d_{n-j,n-k}(z)$ in (4.28):

$$|d_{n-j,n-k}(z)| \leq C\left\{ \frac{(j/n^{1/3})^2 + |\zeta|^2 + 1}{n}\max_{|\theta|<1}|R_\zeta((j+\theta)/n^{1/3},k/n^{1/3})| \right.$$
$$\left. +n^{-1}|j/n^{1/3}|\max_{|\theta|<1}\left|\frac{\partial}{\partial x}R_\zeta((j+\theta)/n^{1/3},k/n^{1/3})\right| \right\}. \qquad (4.29)$$

Similarly

$$((J^{(n)} - zI)R^*)_{n-k,n-k} = -aR^*_{n-k,n-k}(z) + \frac{a}{2}R^*_{n-k+1,n-k}(z) + \frac{a}{2}R^*_{n-k-1,n-k}(z)$$
$$-n^{-2/3}\zeta R^*_{n-k,n-k}(z) - c\frac{k}{n}R^*_{n-k+1,n-k}(z) - c\frac{k-1}{n}R^*_{n-k-1,n-k}(z)$$
$$+r^{(n)}_k R^*_{n-k+1,n-k}(z) - r^{(n)}_{k-1}R^*_{n-k-1,n-k}(z) + \overline{r}^{(n)}_k R^*_{n-k,n-k}(z) \qquad (4.30)$$
$$= \left(-\frac{a}{2}\frac{\partial}{\partial x}R_z(x-0,y) + \frac{a}{2}\frac{\partial}{\partial x}R_z(x+0,y)\right)\Bigg|_{x=y=kn^{-1/3}} + d_{n-k,n-k}(z).$$

The expression in the square brackets is 1 because of equation (3.15), and the remainders $d_{k,k}(z)$ admits the bound

$$|d_{n-k,n-k}(\zeta)| \leq C n^{-1/3}\left\{ (|k/n^{1/3}| + |\zeta| + 1)\max_{|\theta|\leq 1}|R_\zeta((k+\theta)/n^{1/3},k/n^{1/3})| \right\}. \qquad (4.31)$$

We will use the bound

$$||\tilde{D}(z)|| \leq ||\tilde{D}(z)||_1 = \left(\max_j \sum_k |d_{n-j,n-k}| \max_k \sum_j |d_{n-j,n-k}| \right)^{1/2}. \qquad (4.32)$$

First, by using Proposition 3.3, it is easy to show that

$$\max_{|k|\leq 2M}|d_{n-k,n-k}| \leq C \cdot n^{-1/3}\max_{|x|\leq\mu}|x||\psi_+(x,\zeta)||\psi_-(x,\zeta)|,$$

where

$$\mu = Mn^{-1/3}. \qquad (4.33)$$

By using asymptotics (3.19) and (3.20), we find that

$$\max_{|k|\leq 2M}|d_{n-k,n-k}| = O(\mu^{1/2}n^{-1/3}) = O((M/n)^{1/2}). \qquad (4.34)$$

Now we have to estimate $\max_{|j|<M} \sum_{k\neq j} |\tilde{d}_{n-jn-,k}|$, and $\max_{|k|<2M} \sum_{j\neq k} |\tilde{d}_{n-j,n-k}|$. A standard, but tedious analysis, based on (4.29) and Proposition 3.3, shows that the leading contribution to these quantities is due to the expression

$$n^{-1} \max_{|j|<2M} (jn^{-1/3})^2 |\psi_-(jn^{-1/3},\zeta)| \sum_{k=j}^{2M} |\psi_+(kn^{-1/3},\zeta)|$$

which is asymptotically equivalent to

$$n^{-2/3} \max_{|x|\leq\mu} x^2 |\psi_-(x,\zeta)| \int_x^\mu |\psi_+(y,\zeta)| dy \leq C\, n^{-2/3}\mu^2\varepsilon^{-1} = C\,\varepsilon^{-1}n^{-4/3}M^2.$$

Lemma is proved.

Proof of Lemma 3.6. Let us chose

$$M_1 = [n^{1/2}\log^6 n],\ \ \varepsilon = n^{-1/12}\log^{-1} n,\ \ L = \left[C_1\frac{n^{-1/2}\log n}{n^{-2/3}\varepsilon}\right], \tag{4.35}$$

where C_1 is defined in Proposition 4.1. Then, by Proposition 4.1, we have

$$n\int_{a+n^{-2/3}L_0}^\infty \rho_n(\lambda)d\lambda = n\int_{a+n^{-2/3}L_0}^{a+n^{-2/3}\varepsilon L} \rho_n(\lambda)d\lambda + o(1).$$

Besides, similarly to the proof of Lemma 3.2, if we consider

$$\rho_{n-M_1}^{(n)}(\lambda) = (n-M_1)^{-1}\sum_{j=1}^{n-M_1}\psi_j^2(\lambda),$$

(we omitted the superscript (n) in ψ's) then we obtain in view of the same proposition

$$(n-M_1)\int_{a+n^{-2/3}L_0}^{a+n^{-2/3}\varepsilon L}\rho_{n-M_1}^{(n)}(\lambda)d\lambda \leq e^{-C\sqrt{n}}.$$

So

$$n\int_{a+n^{-2/3}L_0}^\infty \rho_n(\lambda)d\lambda = \int_{a+n^{-2/3}L_0}^{a+n^{-2/3}\varepsilon L}\sum_{k=1}^{M_1}\psi_{n-k}^2(\lambda)d\lambda + o(1)$$
$$= \sum_{p=[L_0/\varepsilon]}^{L-1}\int_{a+n^{-2/3}\varepsilon p}^{a+n^{-2/3}\varepsilon(p+1)}\sum_{k=1}^{M_1}\psi_{n-k}^2(\lambda)d\lambda + o(1) = \sum_{p=[L_0/\varepsilon]}^{L-1} I_p + o(1). \tag{4.36}$$

The term I_p of the sum in the r.h.s. admits the bound:

$$
\begin{aligned}
I_p &\leq 2\varepsilon^2 n^{-4/3}\sum_{k=1}^{M_1}\int_{a+n^{-2/3}\varepsilon p}^{a+n^{-2/3}\varepsilon(p+1)}\frac{\psi_{n-k}^2(\lambda)d\lambda}{|\lambda-a-p\varepsilon n^{-2/3}|^2+n^{-4/3}\varepsilon^2}\\
&\leq 2\varepsilon n^{-2/3}\sum_{k=1}^{M_1}|\Im R_{n-k,n-k}^{(n)}(a+n^{-2/3}\zeta_p)|\\
&\leq 2\varepsilon n^{-2/3}\sum_{k=1}^{M_1}|\Im R_{n-k,n-k}^*(a+n^{-2/3}\zeta_p)|\\
&\quad +2\varepsilon n^{-2/3}\sum_{k=1}^{M_1}\sum_{j=-2M_1}^{2M_1}|R_{n-k,n-j}^{(n)}(a+n^{-2/3}\zeta_p)\tilde{D}_{n-j,n-k}(a+n^{-2/3}\zeta_p)|\\
&\quad +2\varepsilon n^{-2/3}\sum_{k=1}^{M_1}\sum_{|j|>2M_1}|R_{n-k,n-j}^{(n)}(a+n^{-2/3}\zeta_p)D_{n-j,n-k}(a+n^{-2/3}\zeta_p)|\\
&= \Sigma_{p1} + \Sigma_{p2} + \Sigma_{p3},
\end{aligned}
\tag{4.37}
$$

where we denote $\zeta_p = \varepsilon(p+i)$ and $D^{(M_1)}$ is defined by (3.25) with M_1 instead of M of (3.12).

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By (3.22) and (3.16),

$$
\begin{aligned}
\Sigma_{p1} &= \frac{4\pi}{\kappa a}\varepsilon n^{-1/3}\sum_{k=1}^{M_1}|\Im\mathrm{Ai}(\kappa\frac{k}{n^{1/3}}+\gamma\zeta_p)\mathrm{Ci}(\kappa\frac{k}{n^{1/3}}+\gamma\zeta_p)| \\
&\leq \frac{4\pi}{\kappa a}\varepsilon n^{-1/3}\sum_{k=1}^{M_1}\left(|\Re\mathrm{Ai}^2(\kappa\frac{k}{n^{1/3}}+\gamma\zeta_p)|+|\Im\mathrm{Ai}(\kappa\frac{k}{n^{1/3}}+\gamma\zeta_p)\mathrm{Bi}(\kappa\frac{k}{n^{1/3}}+\gamma\zeta_p)|\right).
\end{aligned}
\tag{4.38}
$$

Furthermore, by the Schwartz inequality 3.4, we get

$$
\begin{aligned}
\Sigma_{p2} &\leq C\,\varepsilon n^{-2/3}\sum_{k=1}^{M_1}\sum_{j=-2M_1}^{2M_1}|R^{(n)}_{n-k,n-j}D_{n-j,n-k}| \\
&\leq C\,\varepsilon\left[n^{-4/3}\sum_{k=1}^{2M_1}\sum_{j=-2M_1}^{2M_1}|R^{(n)}_{n-j,n-k}|^2\right]^{1/2}\left[\sum_{k=1}^{M_1}\sum_{j=-2M_1}^{2M_1}|D_{n-j,n-k}|^2\right]^{1/2}.
\end{aligned}
\tag{4.39}
$$

For the first factor we use Lemma 3.4 in which M of (3.12) is replaced by $2M_1$ of (4.35) and ε is also given by (4.35). Since in this case we have by (3.28) $||D^{(2M_1)}|| = O(n^{-1/4}\log^{13}n)$, the conditions of the lemma are satisfied and we obtain in view of (3.22)

$$
n^{-4/3}\sum_{k=1}^{2M_1}\sum_{|j|\leq 2M_1}|R^{(n)}_{n-k,n-j}|^2 \leq 3n^{-2/3}\sum_{k=1}^{2M_1}\sum_{|j|\leq 4M_1}\left|R_{\zeta_p}\left(\frac{k}{n^{1/3}},\frac{j}{n^{1/3}}\right)\right|^2 + o(n^{-1}).
$$

Besides, we have in view of (4.29):

$$
\begin{aligned}
\sum_{k=1}^{M_1}\sum_{j=-2M_1}^{2M_1}|D_{n-k,n-j}|^2 &\leq C\frac{M_1^4}{n^{10/3}}\sum_{k=1}^{M_1}\sum_{j=-2M_1}^{2M_1}\left|R_{\zeta_p}\left(\frac{k}{n^{1/3}},\frac{j}{n^{1/3}}\right)\right|^2 + C\frac{M_1^2}{n^{4/3}}\sum_{k=1}^{M_1}\left|R_{\zeta_p}\left(\frac{k}{n^{1/3}},\frac{j}{n^{1/3}}\right)\right|^2 \\
&\quad + C\frac{M_1^2}{n^{8/3}}\sum_{k=1}^{M_1}\sum_{j=-2M_1}^{2M_1}\left|\frac{\partial}{\partial x}R_{\zeta_p}\left(\frac{k}{n^{1/3}},\frac{j}{n^{1/3}}\right)\right|^2.
\end{aligned}
$$

Furthermore, by using representations (3.22) and (3.20), we obtain for any $0\leq k\leq M_1$, $|j|>2M_1$ and sufficiently large n

$$
|D^{(M_1)}_{n-j,n-k}(a+n^{-2/3}\zeta_p)| \leq n^{1/3}e^{-C\varepsilon(M_1 n^{-1/3})^{1/2}}\exp\left\{-C\frac{||j|-2M_1|^{1/2}}{n^{1/4}\log n}\right\}.
\tag{4.40}
$$

Here we took into account that $|\Re\zeta_p|\leq L\varepsilon << M_1$. Thus, $\Sigma_{p3}=o(n^{-1})$.

The above bounds and a bit tedious but routine calculations, based on Proposition 3.3 and the Euler-Mclaurin summation formula [1] yield the r.h.s. of (3.32)

The same arguments can be applied also to the other endpoints of the spectrum. Thus, Lemma 3.6 is proved.

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